WS 2016/17 Dr. Milena Pabiniak
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Symplectic Geometry

Homework 11

Exercise 1. (10 points)
Define $f\colon \mathbb{C}^n \to \mathbb{R}$ by $f(z) = log(1+|z|^2)$ and let

$$[h_{jk}] = \left[\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}\right].$$

- Show that f is strictly plurisubharmonic, i.e. that $[h_{jk}]$ is positive definite.
- (Bonus; look for a short solution) Prove that

$$\det[h_{jk}] = (1+|z|^2)^{-(n+1)}.$$

Exercise 2. (10 points)
In this problem we go back to the Kodaira Thurston example of a manifold which is complex and symplectic but not Kähler. Recall the definition of Γ (from the lecture or from "Introduction to Symplectic Geometry" by Ana Cannas da Silva, page 102). Let $\widetilde{\Gamma}$ be the group generated by $\gamma_1, \gamma_2, \gamma_3$.

- Find $\mathbb{R}^4/\widetilde{\Gamma}$ and observe that $\mathbb{R}^4/\Gamma$ is the quotient of $S^1 \times S^1 \times \mathbb{R} \times S^1$ by the action of $\mathbb{Z}$ given by
$$j(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2 + jy_2, y_1 + j, y_2).$$
- Find the commutator $[\Gamma, \Gamma]$ and show that $\Gamma/[\Gamma,\Gamma]$ is isomorphic to $\mathbb{Z}^3$.

Recall from the lecture that from the above fact one can deduce that $\mathbb{R}^4/\Gamma$ is not Kähler. Indeed, let $p\colon \mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ denote the projection map. By the Theorem 1.40 in "Algebraic Topology" of A. Hatcher we have that $\pi_1(\mathbb{R}^4/\Gamma) \,/\, p_*(\pi_1(\mathbb{R}^4)) = \Gamma$. Therefore $H_1(\mathbb{R}^4/\Gamma; \mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ is of rank 3 and thus the symplectic manifold $\mathbb{R}^4/\Gamma$ is not Kähler as a theorem of Hodge implies that for Kähler manifolds odd Betti numbers are even.

Exercise 3. (10 points)
Show that for a matrix group G (i.e. a subgroup of $GL(n,\mathbb{R})$ for some n)

$$[X,Y] = XY - YX, \quad \frac{d}{dt} Ad_{\exp tX} Y\,|_{t=0} = [X,Y], \quad Ad_{gh} = Ad_g \circ Ad_h, \quad Ad^*_{gh} = Ad^*_g \circ Ad^*_h$$

for any $X, Y \in \mathfrak{g} = Lie(G)$ and $g, h \in G$. (See "Introduction to Symplectic Geometry" by Ana Cannas da Silva, page 131).

Exercise 4. (10 points)
Exercise 1 from Homework 16 in "Introduction to Symplectic Geometry", page 132.

Hand in: Thursday January 19 th
in the exercise session
in Übungsraum 1, MI